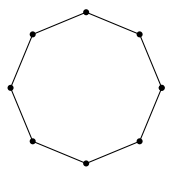
3201. Prove that quadratic graphs $y = x^2 + ax + b$ and $y = -x^2 + cx + d$, for any constants $a, b, c, d \in \mathbb{R}$, may be transformed onto each other by a rotation.

3202. A heavy chain consisting of n links each of mass m kg is held at one end, hanging freely. A further tension is then applied, and the chain accelerates upwards at a ms^{-2}.

(a) With $k = 1$ topmost, find an expression for the contact force between the k^{th} and $(k+1)^{\text{th}}$ links, in terms of a, k, m and n.

(b) Hence, show that, when a heavy uniform rope hangs vertically, the tension decreases linearly down the rope.

3203. Four distinct vertices are chosen at random from among those of a regular octagon.



Find the probability that these four form a square.

3204. A family of straight lines is defined, for $k \in \mathbb{R}$, by

$$\left(k^2 - 1\right) y = 2kx.$$

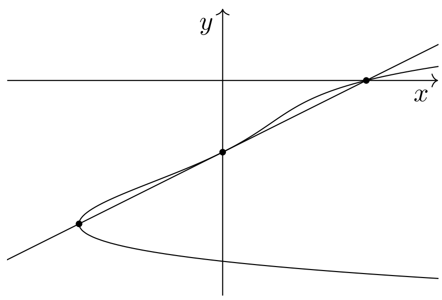
Show that this family contains all of the straight lines through the origin of the (x, y) plane.

3205. Show that the equation $8^x + 1 = 2^{x+1}$ has solution set $\{0, \log_2(\sqrt{5} - 1) - 1\}$.

3206. A quartic curve is given as

$$x = 16y^4 + 32y^3 + 20y^2 + 8y + 2.$$

A tangent is drawn to the curve at $B : (0, -1/2)$. The tangent re-intersects the curve at A and C.



Show that $|AB| = |BC|$.

3207. (a) Show that

$$\sin x \left(\tfrac{1}{2} \sin 2x - 1\right) \equiv \cos x - \sin x - \cos^3 x.$$

(b) Hence, solve $\cos x = \sin x + \cos^3 x$, giving all roots $x \in [0, 2\pi)$.

3208. You are given that the variables x, y, z satisfy the differential equation

$$\frac{dz}{dx} \propto \frac{dx}{dy}.$$

Verify that $z \propto y^3 \propto x^{-1}$ is a solution.

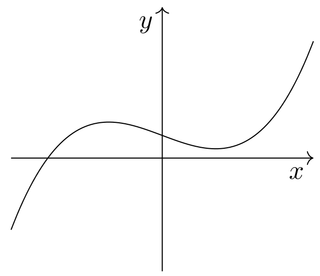
3209. An algebraic fraction is given as

$$\frac{x^4 + 3x^2}{x + 1}.$$

Write this as the sum of a polynomial and a proper algebraic fraction.

3210. Show that $x + 27y = 8$ is tangent to $y = \dfrac{x}{(x+1)^2}$.

3211. Show that $y = (3x + 7)^3 + 1$ could not possibly be the equation generating the following graph:



3212. The lengths of adult sablefish are modelled with a normal distribution $L \sim N(60, 100)$, where the units are centimetres.

(a) Find the probability that a sablefish is over 70 cm in length.

(b) Find the probability that, when two sablefish are examined, at least one of them is over 70 cm in length.

3213. State, with a reason, whether these hold:

(a) $a + b \in \mathbb{Z} \implies a, b \in \mathbb{Z}$,

(b) $ab \in \mathbb{Z} \implies a, b \in \mathbb{Z}$.

3214. Stating any assumptions you make, show that, when β is small,

$$\sin(\alpha + \beta) \approx \sin\alpha + \beta \cos\alpha.$$

3215. A cubic graph has equation

$$y = (x + b)^3 + (x + b)^2 + x + b + 1.$$

A tangent is drawn to the cubic at the point with x coordinate $2 - b$. Find the y coordinate of the point at which this tangent line re-intersects the curve.

3216. Show that $\displaystyle\int_0^1 \frac{\sqrt{x}}{1+\sqrt{x}}\,dx = \ln 4 - 1$.

3217. From a committee of fifteen people, a chairperson, two secretaries, three adjutants and four liaisons are chosen. Find the number of different ways in which this can be done.

3218. Prove the identity $\tan\left(\theta + \frac{\pi}{4}\right) \equiv \sec 2\theta + \tan 2\theta$.

3219. The equations $\mathrm{f}(x) = 0$ and $\mathrm{g}(x) = 0$, where f and g are polynomial functions of the same degree n, have the same solution set S, which consists of n distinct real numbers. The equation $\mathrm{f}(x) = \mathrm{g}(x)$ is denoted E. State, with a reason, whether the following claims hold:

(a) "E has solution set S",

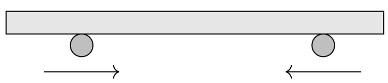
(b) "the solution set of E contains S",

(c) "the solution set of E is a subset of S".

3220. Let $a, b \in (0, 1)$ be two irrational numbers $a < b$, which differ when truncated to 1 decimal place. Prove that there is at least one rational number in the interval (a, b).

3221. Evaluate $\displaystyle\sum_{k=1}^{4n} \sin 90k°$, for $n \in \mathbb{N}$.

3222. A practical method of finding the centre of mass of a non-uniform rod is as follows. Firstly, balance the rod on two identical low-friction supports such as a pair of pencils. Then, slowly, move the pencils towards each other.



The supports meet at the centre of mass. Explain, with reference to moments and the coefficient of friction model, why this works.

3223. A computer scientist sets up a function to answer the following question: "For a polynomial function f, how many asymptotes of the form $x = k$ does the graph $y\,\mathrm{f}(x) = 1$ have?" Give

(a) a suitable domain,

(b) a suitable codomain,

(c) the range, with the domain given.

3224. A *biquadratic* graph has the form $y = ax^4 + bx^2 + c$. Prove that, if a biquadratic graph has two local minima, then they must both be global minima.

3225. Prove that, for any set of data, $\sum x^2 \geq n\bar{x}^2$.

3226. A metals company is trying to increase efficiency. Its bolts, which must fit into nuts of size 5 mm, are currently produced with lengths, in mm, modelled by $N\big(4.96, 0.02^2\big)$.

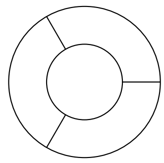
(a) Find the probability that a randomly selected bolt will fail to fit a 5 mm nut.

(b) Calculate F, the expected number of failures in a box of 500 bolts.

(c) Determine, to 4dp, the reduced value of the mean that would bring F down to 5.

(d) Find instead, to 4dp, the reduced value of σ that would bring F down to 5.

3227. The regions of the following diagram are randomly coloured, each red, yellow, green or blue. Multiple regions can be the same colour.



(a) Find the probability that every colour appears on the map.

(b) Find the probability that exactly two colours appear, given that the central region is blue.

3228. In this question, do not use a polynomial solver. The question concerns the factorisation of

$$\mathrm{f}(x) = 90x^3 - 1121x^2 + 3348x - 2737.$$

(a) Find $\mathrm{f}'(x)$.

(b) Set up Newton-Raphson for $\mathrm{f}(x) = 0$, and hence suggest a factor of $\mathrm{f}(x)$. Use the starting point $x_0 = 1$.

(c) Factorise $\mathrm{f}(x)$ fully.

3229. Two circles have equations

$$x^2 + y^2 = 1,$$
$$(x-4)^2 + y^2 = 9.$$

There are two lines L_1 and L_2 which are tangent to both circles but do not pass between them. Show that these meet each other at $(-2, 0)$.

3230. Prove that, for any linear function $\mathrm{f} : \mathbb{R} \mapsto \mathbb{R}$, and any arithmetic progression a, b, c, d,

$$\int_a^d \mathrm{f}(x)\,dx = 3\int_b^c \mathrm{f}(x)\,dx.$$

3231. True or false?

(a) $y = x^3 - x$ has a point of inflection,

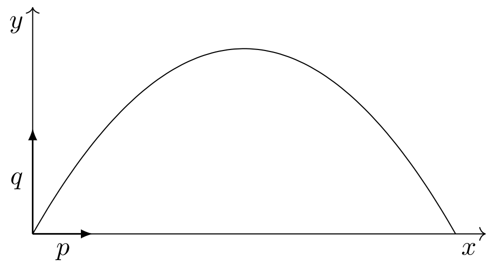
(b) $y = x^4 - x^2$ has a point of inflection,

(c) $y = x^5 - x^3$ has a point of inflection.

3232. Determine the number of points (x, y) in the set

$$\{(x, y) \in \mathbb{Z}^2 : 20 < x^2 + y^2 < 30\}.$$

3233. A projectile is launched from an origin at ground level. It has initial horizontal velocity p and initial vertical velocity q, where $p, q > 0$.



(a) Find the equation of the trajectory, writing the coefficients in terms of p, q and g.

(b) Hence, show that the range is $\dfrac{2pq}{g}$.

(c) When the projectile lands, it is travelling at an angle θ below the horizontal. Find θ, in terms of p and q.

3234. Two sinusoidal curves are given as

$$y = 2\sin x + 4\cos x,$$
$$y = 2 + \sin x + 3\cos x.$$

Show that these curves do not intersect.

3235. Two random variables, which are independent, each follow the standardised normal distribution $Z_1, Z_2 \sim N(0, 1)$. Find the probability that their sum exceeds 1.

3236. Find the exact area of the region R of the (x, y) plane which satisfies all three of the inequalities $x^2 + y^2 \le 5$, $x + y \ge 1$, and $x - y \le 1$.

3237. Line L is given as $y - b = \tan\theta(x - a)$, for some constants a, b and θ. Find, in similar form, the equation of the reflection of L in

(a) the line $x = a$,

(b) the line $y = b$.

3238. A curve is given as $x^2 + y^2 = \frac{5}{12}x^3 + 1$.

(a) Explain why, in the vicinity of the y axis, this graph approximates a unit circle.

(b) Find, to 3sf where appropriate, the coordinates of the four points at which the tangent is parallel to the x axis and the one at which it is parallel to the y axis.

(c) Hence, sketch the curve.

3239. Determine the number of roots of the equation

$$(x^2 + x - 17)(x^6 + x^3 - 17) = 0.$$

3240. In a game of darts, players A and B are taking turns aiming for the inner bull. If either hits the inner bull (probability 0.1) they win; if either hits the outer bull (probability 0.2), they lose. Any other score and the game continues. Player A throws first. Find the probability that the game ends with an inner bull.

3241. Simultaneous equations are given as

$$0 = 2\cos(\theta - \phi) - 2\cos(\theta + \phi) - 1,$$
$$0 = \sin\theta - \cos\phi.$$

Find the values of ϕ in $[0, 2\pi)$ for which there are (θ, ϕ) solution points.

3242. State whether each of the following implications holds, for $a, b \in \mathbb{R}$. If not, give a counterexample.

(a) $a > b \implies ab > b^2$,

(b) $a > b \implies e^a > e^b$,

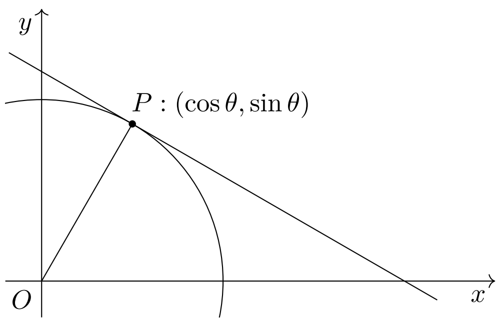
(c) $a > b \implies a^3 - a > b^3 - b$.

3243. Sketch $x^2y + y - 1 = 0$.

3244. A unit circle is drawn, with P defined as the point $(\cos\theta, \sin\theta)$. Consider four lines:

① the tangent at P,
② the radius OP,
③ the x axis,
④ the y axis.

Together, these form three right-angled triangles.



For each triangle, give the trigonometric identity generated by applying Pythagoras's theorem.

3245. Two small asteroids have positions, relative to an origin, given in metres by

$$\mathbf{r}_1 = \begin{pmatrix} 20 + 2t \\ 35 - t \\ 25 \end{pmatrix}, \quad \mathbf{r}_2 = \begin{pmatrix} 15 + t \\ -45 + t \\ 35 + 2t \end{pmatrix}.$$

Show that, at their closest approach, the asteroids are around 67 metres from each other.

3246. A curve is defined by $y = \dfrac{x}{x^2+1}$.

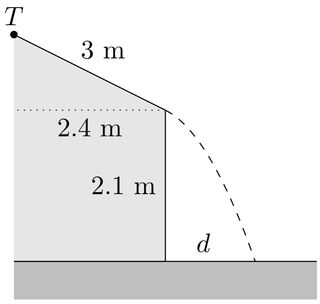
(a) Find the stationary points of the curve.

(b) Show that the curve has a point of inflection at the origin.

(c) By also considering the behaviour as $x \to \pm\infty$, sketch the curve.

3247. Solve the equation $(a^2 - 1)^3 - a^2 + 1 = 0$.

3248. A tile becomes detached, and falls from a sloped roof, as depicted below. The tile, which may be modelled as a particle, begins at rest, at the point marked T. The coefficient of friction between the tile and the roof is modelled as $\mu = \frac{1}{2}$.



(a) Draw a force diagram for the tile, giving any relevant angles.

(b) Determine the horizontal and vertical speeds with which the tile leaves the roof.

(c) Find the horizontal distance d away from the building at which the tile lands.

3249. A differential equation is given as

$$\left(\frac{dy}{dx} - 1\right)\left(\frac{dy}{dx} - 2\right) = 0.$$

Determine whether each of the following graphs satisfies the above differential equation:

(a) $y = x + 3$

(b) $y = 2x + 1$

(c) $y = (x + 3)(2x + 1)$

3250. Solve $\sin t + \left(\sqrt{2} - 1\right)\cos t = 1$ for $t \in [-\pi, \pi)$. Give your answers in exact form.

3251. Determine the range of $x \mapsto e^{\sec x}$.

3252. A graph is given as $y = \ln(x^2 + 1)$.

(a) Find the coordinates of any stationary points and points of inflection.

(b) Hence, sketch the curve.

3253. A triangle ABC has vertices at $\mathbf{a}, \mathbf{b}, \mathbf{c}$. Point P is then defined, for positive constants $\lambda_1, \lambda_2, \lambda_3$ with $\sum \lambda_i = 1$, by

$$\mathbf{p} = \lambda_1 \mathbf{a} + \lambda_2 \mathbf{b} + \lambda_3 \mathbf{c}.$$

(a) The point A is translated by $k\overrightarrow{AB} + (1-k)\overrightarrow{AC}$. Describe the set of points attainable by such a translation.

(b) Express $\overrightarrow{AP}$ in the form $k_1\overrightarrow{AB} + k_2\overrightarrow{AC}$.

(c) Hence, prove that P is inside triangle ABC.

3254. A function f has instruction

$$\mathrm{f}(x) = \frac{1}{1+\sin x} + \frac{1}{1-\sin x} - 2.$$

Simplify $\mathrm{f}(x)$ fully.

3255. (a) By proposing an answer of the form $a\sqrt{3} + b$, determine exactly the square roots of $8 - 4\sqrt{3}$.

(b) For an acute angle ϕ, $\cot\phi = 2 + \sqrt{3}$. Show that $\sec\phi = \sqrt{6} - \sqrt{2}$.

3256. A differential equation is given as

$$\frac{dy}{dx} = \frac{1-y}{1-x}.$$

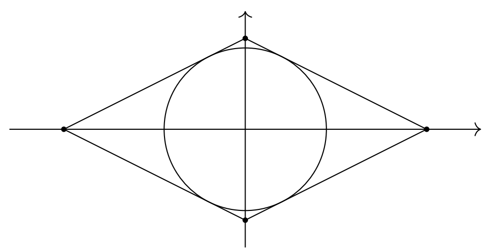
Show that any polynomial solution curve of the DE must pass through the point $(1, 1)$.

3257. The following claim is made:

"The combination of two samples of the same size, one with strong positive correlation and one with strong negative correlation, cannot yield strong positive correlation."

Sketch a counterexample, in the form of a scatter diagram, to disprove the claim.

3258. A rhombus, centred at the origin of an (x, y) plane, is tangent to the unit circle at four points. Its sides are at inclination θ to the x axis.



Show that the perimeter is given by $P = 8\operatorname{cosec} 2\theta$.

3259. (a) Sketch $x = \sin^2 y$.

(b) Hence, sketch $\sqrt{x} = \sin y$.

3260. A number $k \in \mathbb{N}$ has prime factorisation

$$k = p_1^{\alpha_1} \times p_2^{\alpha_2} \times \ldots \times p_n^{\alpha_n}.$$

Evaluate $\displaystyle\sum_{i=1}^{n} \alpha_i \log_k p_i$.

3261. A rigid object is in equilibrium under the action of three forces in an (x, y) plane, two of which are

$$10\mathbf{i} - 10\mathbf{j} \text{ N at } (0, 6),$$
$$20\mathbf{j} \text{ N at } (3, 1).$$

Determine the magnitude of the third force, and show that its line of action passes through O.

3262. A model relating variables x and y is given, for some constants $a \in \mathbb{R}$ and $n \in \mathbb{N}$, as

$$y = ax^n.$$

Prove that, according to the model, the following expression is constant:

$$\frac{\log_2 y - \log_2 a}{\log_2 x}.$$

3263. Two sequences are defined ordinally by

$$a_n = 20n - n^2,$$
$$b_n = 500 - 40n + n^2.$$

Determine the single value of $p \in \mathbb{N}$ and the single value of $q \in \mathbb{N}$ for which $a_p = b_q$.

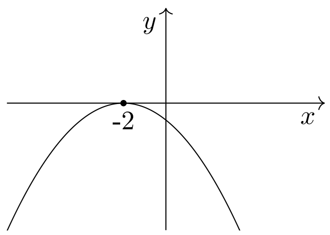
3264. In this question, f is a polynomial function and $\mathrm{f}(x)$ contains no even powers of x. Note that zero is an even number.

State, with a reason, whether the curve $y = \mathrm{f}(x)$ necessarily intersects the following curves:

(a) $y = \mathrm{f}(-x)$,
(b) $y = -\mathrm{f}(x)$,
(c) $y = -\mathrm{f}(-x)$.

3265. A Pythagorean triple gives a right-angled triangle with sides of integer length. Prove that the area of such a triangle is also an integer.

3266. The graph below is of $y = \mathrm{f}'(x)$, for some cubic function f defined over $\mathbb{R}$.



(a) Find the set of inputs for which f is concave.
(b) Show that $\mathrm{f}(x) = 0$ has exactly one root.

3267. Two vectors $\mathbf{a}$ and $\mathbf{b}$ combine to give perpendicular unit vectors $\mathbf{i}$ and $\mathbf{j}$ in the following manner:

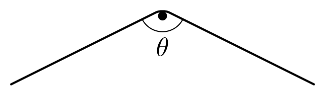
$$\mathbf{a} + \mathbf{b} = \mathbf{i},$$
$$3\mathbf{a} + 2\mathbf{b} = \mathbf{j}.$$

Express $\mathbf{a}$ and $\mathbf{b}$ as column vectors.

3268. In a computer program, three variables X_1, X_2, X_3 are binary, taking values in the set $\{0, 1\}$. To begin with, they are all set to zero. A random number from the set $\{0, 1, 2, 3\}$ is then chosen, which gives the number of X_i variables available for change. Each of these, with probability $\frac{1}{2}$, may then change to 1. Find the probability that

(a) three variables change to 1,
(b) two variables change to 1.

3269. A light rope is passed over a smooth, fixed peg. The sections of rope either side of the peg make a non-reflex angle θ with each other, and the tension in the rope is T.



Show that the force F exerted on the peg by the rope is given by

$$F = T\sqrt{2(1 + \cos\theta)}.$$

3270. The locus of the equation $x^2y^2 - 24 = 10xy$ is made up of two hyperbolae. Find their equations and hence sketch the curve.

3271. In this question, an operation $\wedge$ is defined, on two functions, as

$$\mathrm{f}(x) \wedge \mathrm{g}(x) = \sqrt{\mathrm{f}(x)\,\mathrm{g}(x)}.$$

(a) State, with a reason, whether it is always true that $\mathrm{f}(x) \wedge \mathrm{f}(x) = \mathrm{f}(x)$.
(b) Sketch the following graphs, over their largest possible real domains:
   i. $y = x \wedge |x|$,
   ii. $y = x^2 \wedge x^4$.

3272. Prove, by contradiction, that there is no smallest positive irrational number.

3273. Assuming the chain and product rules, prove the quotient rule

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}.$$

3274. An *amortised* mortgage is one repaid in constant, regular instalments. P is the initial amount owed, c is the amount paid off at the end of every month and r is the monthly interest rate. Show that T_2, the total outstanding after two months, is given by

$$T_2 = P(1 + r)^2 - c(2 + r).$$

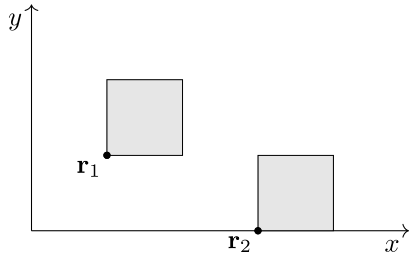
3275. For $Z \sim N(0,1)$, find $\mathbb{P}(Z^3 - 1 > Z)$ to 3sf.

3276. You are given that, for $\theta \in [0, \pi]$,

$$\cos\theta = \frac{x^2 - y^2}{x^2 + y^2}.$$

Find $\sin\theta$ in simplified terms of x and y.

3277. Two squares of unit side length are moving in the (x, y) plane. Their sides are parallel to the axes.



The lower left-hand vertices of the squares have position vectors given, in terms of time t, by

$$\mathbf{r}_1 = \begin{pmatrix} t \\ t \end{pmatrix}, \quad \mathbf{r}_2 = \begin{pmatrix} 4-t \\ 0 \end{pmatrix}.$$

Show that the squares never overlap.

3278. An antibody test for a particular disease is being offered to patients at a clinic.

- Among those who have had the disease, the probability that an individual agrees to take the test is 15%.
- Among those who have not had the disease, this probability is only 10%.

The trial shows that 30% of those who took the test had had the disease.

(a) Find the proportion of the population who have had the disease.

(b) Explain why this value is only an estimate.

3279. Solve $(e^{2x} - 1)^3 - (e^x - 1)^3 = 0$.

3280. The area under a bell curve is being estimated by the trapezium rule. The curve is $y = \mathrm{f}(x)$, for

$$\mathrm{f}(x) = e^{-\frac{1}{2}x^2}.$$

(a) Evaluate $\mathrm{f}''(0)$, $\mathrm{f}''(1/2)$ and $\mathrm{f}''(1)$.

(b) Hence, explain why, given the same number of strips, an estimate for $0 \leq x \leq 1/2$ will produce a greater percentage error than an estimate for $1/2 \leq x \leq 1$.

3281. A quadratic equation $ax^2 + bx + c = 0$ has distinct real roots at $x = p, q$. Give a quadratic equation, with coefficients in terms of a, b, c, whose roots are at $x = -2p, -2q$.

3282. Sketch $(x - 2y)^2 + (x - 2y) = 0$.

3283. A jar contains r red and b blue counters. From the jar, two counters are taken out at random. You are given that the probability of two blues is $3/20$ and the probability of one red, one blue is $1/2$. Find r and b.

3284. A, B, C are three statements, with their negations denoted by a prime. State, with a reason, whether the following argument is valid:

"The implications $A \implies B$ and $C' \implies B'$ are together sufficient to prove that $A \implies C$."

3285. On an (x, y) plane with unit vectors $\mathbf{i}$ and $\mathbf{j}$, points P and Q are defined in terms of a parameter $t \in [0, 1]$. Their position vectors are

$$\mathbf{p} = t\mathbf{i},$$
$$\mathbf{q} = (1 - t)\mathbf{j}.$$

(a) Sketch the line segment PQ.

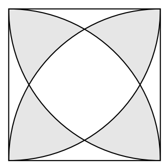
(b) The point closest to O on PQ is labelled X. Find, in terms of t, the coordinates of X.

(c) Show that $|OX| = \dfrac{t(1-t)}{\sqrt{2t^2 - 2t + 1}}$.

(d) Explain how you know that this expression for $|OX|$ is symmetrical around $t = \frac{1}{2}$.

3286. "The curves $y = x^3 + p$ and $x = y^3 + p$ are tangent for exactly two values $p \in \mathbb{R}$." True or false?

3287. A pattern is constructed from a square and four quarter circles, as depicted below:



Show that the area shaded is $\frac{1}{3}\pi - 4 + 2\sqrt{3}$.

3288. Show that, for small angles x in radians,

$$\frac{\cos 3x - 1}{x \sin 4x} \approx -\tfrac{9}{8}.$$

3289. There is an error in the following statement. Find and correct it.

"In a binomial hypothesis test with $H_1 : p > p_0$, if c is the critical value at the 2% level, then

$$\mathbb{P}(X \geq c) < 0.01 < \mathbb{P}(X \geq c - 1)".$$

3290. The equations $3x + 4y = k$, $x^2 + y^2 = 1$ do not have any real simultaneous solutions. Find the set of possible values of k.

3291. A sequence, with first term a, is defined iteratively by $u_{n+1} = ru_n$. Its partial sums S_n are defined by adding the first n terms of the sequence together:

$$S_n = u_1 + u_2 + ... + u_n.$$

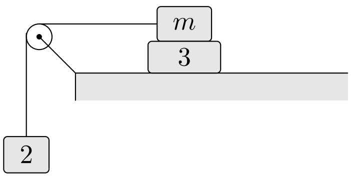
(a) Prove that the nth term of the sequence is given by $u_n = ar^{n-1}$.

(b) By calculating the quantity $S_n - rS_n$, prove the geometric series formula

$$S_n = \frac{a(1-r^n)}{1-r}.$$

3292. Solve $(x^2-4)^4 - (x^2-4)^3 = 0$.

3293. A pulley system is set up on a table as depicted. Masses are given in kg. The pulley is light and smooth, and the string is light and inextensible. The coefficient of friction at the upper surface of the lower block is $2/5$ and at its lower surface is $1/5$.



(a) Assuming that the lower block doesn't move, find the set of values of m for which the upper block does move.

(b) Assuming that the upper block does move, find the set of values of m for which the lower block doesn't move.

(c) Hence, find the set of values of m for which all three blocks move.

3294. By defining $u = \cos x$, show that the rate of change of $\sin x$ with respect to $\cos x$ is $-\cot x$.

3295. In each case, find the value of the limit:

(a) $\displaystyle\lim_{x\to 1} \frac{x^2-1}{x^2-x}$,

(b) $\displaystyle\lim_{x\to 1} \frac{x^3-1}{x^3-x}$.

3296. A large population is modelled by the distribution $N(40, 10^2)$. Values sampled from this population are represented by variables X_i. Find the following probabilities:

(a) $\mathbb{P}(X_1 > 50)$,

(b) $\mathbb{P}(X_1 + X_2 > 100)$.

3297. The lines $y = (2 \pm \sqrt{3})x$ are drawn. A third line is to be added, to form an equilateral triangle. Show that this line must have the form $x + y = k$.

3298. Prove that $y = \frac{1}{2}x - 1 \implies y < x^2 - x$.

3299. An isosceles triangle has area 168, perimeter 64.

(a) Labelling the side lengths as $(2a, b, b)$, set up equations in a and b.

(b) Show that $a = \dfrac{168}{\sqrt{1024-64a}}$.

(c) Using fixed-point iteration, find the lengths of the sides of the triangle.

3300. Solve for x in $2^{\log_2 x} + 2^{\log_4 x} = 12$.

——— End of 33rd Hundred ———